affine stacks

Defn / Notation: Fix base ring k.

$\mathrm{coSCR}_k = \infty$-cat. of cosimplicial comm. ring / k.

$\mathrm{St}_k = \infty$-cat of stacks in spaces / $\{k\text{-alg.}\}$

$\subseteq \mathrm{Shv}^{fpqc}_{\mathrm{Spaces}}(k\text{-alg}^{op})$

$$\mathrm{Spec}^{\Delta} : \ {}_{\Delta}\mathrm{SCR}_k^{op} \underset{\perp}{\rightleftarrows} \mathrm{St}_k \ : R\Gamma(-,\mathcal{O}) = C^*_{\Delta}(-,\mathcal{O})$$

$$\left(A_0 \rightrightarrows A_1 \substack{\rightarrow\\ \rightarrow\\ \rightarrow} A_2 \cdots\right) \longmapsto \left(\cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathrm{Spec}\, A_1 \rightrightarrows \mathrm{Spec}\, A_0\right)$$

essential image of Spec^{Δ} is called <u>affine stacks</u>.

Lemma: {affine stacks} closed under limit.

Construction: fix a base S. $\boxed{K(H,1)}$

H gp scheme $/S$, form stack BH $\left(\begin{array}{ccc} H & \to & * \\ \downarrow & & \downarrow \\ * & \to & BH \end{array}\right)$

$$\left(\cdots \begin{array}{c}\to\\\to\\\to\\\to\end{array} H \times_S H \overset{pr_1}{\underset{pr_2}{\overset{m}{\rightrightarrows}}} H \rightrightarrows S \right) =: \to BH$$

given test alg R

$$\begin{array}{ccc} H(\operatorname{Spec} R) & \longrightarrow & pt \\ \downarrow & \lrcorner & \downarrow \\ pt & \longrightarrow & BH(\operatorname{Spec} R) \end{array}$$

$,*$

$$\Longrightarrow \pi_{i+1}(BH(\operatorname{Spec}(R))) = \pi_i(H(\operatorname{Spec}(R))) \quad \forall i \geq 0$$

$$\pi_0 BH(\operatorname{Spec}(R)) = \{H\text{-torsors}\}.$$

if H is abelian.

$H \times H \xrightarrow{m} H$ is a gp hom.

$\rightsquigarrow B(H \times H) \xrightarrow{B(m)} BH$

$$K(H,2)$$

$$\begin{tikzcd}
BH \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & B(BH)
\end{tikzcd}$$

$$\begin{tikzcd}
\Big(\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & BH \times BH \arrow[r, shift left=2, "pr_1"] \arrow[r, "B(m)"] \arrow[r, shift right=2, "pr_2"'] & BH \arrow[r, shift left] \arrow[r, shift right] & S \Big) \arrow[r] & B(BH) \\
\cdots & S \arrow[u] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & S \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & S \arrow[u] & S \arrow[u] \\
\cdots & H \times H \arrow[u, shift left] \arrow[u, shift right] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & H \arrow[u, shift left] \arrow[u, shift right] \arrow[r, shift left] \arrow[r, shift right] & S \arrow[u, shift left] \arrow[u, shift right] & H \arrow[u, shift left] \arrow[u, shift right] \\
& H^4 \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & H \times H \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] \arrow[r, shift left] \arrow[r, shift right] & \vdots \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & H^4 \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & H^9 \cdots \Big) \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3]
\end{tikzcd}$$

Defn: Recall in Emanuel's talk

$$0 \longrightarrow H \longrightarrow W^{Fil} \xrightarrow{\text{Frob}-[\lambda^{p-1}]} W^{Fil} \longrightarrow 0 \quad / [\mathbb{A}^1/\mathbb{G}_m]_{\mathbb{Z}_{(p)}}$$

$$S^1_{Fil} := BH \longrightarrow [\mathbb{A}^1/\mathbb{G}_m].$$

$$BS^1_{Fil} := B(BH) \longrightarrow [\mathbb{A}^1/\mathbb{G}_m].$$

Prop: S^1_{Fil} & BS^1_{Fil} are relatively affine over $[\mathbb{A}^1/\mathbb{G}_m]_{\mathbb{Z}_{(p)}}$

Hence $\left(\mathcal{S}^1_{\mathrm{Fil}}\right)^u$ & $\left(B\mathcal{S}^1_{\mathrm{Fil}}\right)^u$ are affine over $\mathbb{Z}_{(p)}$

and $\left(\mathcal{S}^1_{\mathrm{Fil}}\right)^{gr}$ & $\left(B\mathcal{S}^1_{\mathrm{Fil}}\right)^{gr}$ relatively affine over $B\mathbb{G}_{m\,\mathbb{Z}_{(p)}}$.

$/\mathbb{Z}_{(p)}$.

$$0 \longrightarrow \ker(G) \longrightarrow W \times \mathbb{A}^1 \xrightarrow{G} W \times \mathbb{A}^1 \longrightarrow 0.$$

$$\begin{tikzcd}
B(\ker(G)) \arrow[r] \arrow[d] \arrow[dr, phantom, "T"] & BW \times \mathbb{A}^1 \arrow[d, "B(G)"] \\
\mathbb{A}^1 \arrow[r] & BW \times \mathbb{A}^1
\end{tikzcd}$$

need to show BW is affine.

$$W \xrightarrow{\cong} \lim_n W_n \rightsquigarrow BW \xrightarrow{(*)} \lim_n BW_n.$$

$$K(W,1) \qquad \lim_n K(W_n, 1)$$

claim: (1) BW_n are affine $\forall n$.

✓ (2) $(*)$ is an equivalence.

$$W \xrightarrow{\sim} \lim W_n \to \pi_1 \quad ✓$$

for π_0: $R^1\lim W_n = 0$. ✓

$$0 \longrightarrow \mathbb{G}_a \longrightarrow W_n \xrightarrow{\text{res}} W_{n-1} \longrightarrow 0$$

$$\begin{tikzcd}
W_n \arrow[r] \arrow[d] & * \arrow[d] & & BW_n \arrow[r] \arrow[d] & * \arrow[d] \\
W_{n-1} \arrow[r] & K(\mathbb{G}_a, 1) \arrow[r, phantom, "\rightsquigarrow"] & {} & BW_{n-1} \arrow[r] & K(\mathbb{G}_a, 2)
\end{tikzcd}$$

(Toën: $K(\mathbb{G}_a, n)$ is affine).

in fact, the proof works for $K(H, n)$ $\forall n$.

Rmk: $\quad S^1_{Fil} \leftarrow \left(* \leftleftarrows H \Lleftarrow H \times H \ \cdots \right)$

everything in sight is flat $/ \, [\mathbb{A}^1/\mathbb{G}_m]_{\mathbb{Z}_{(p)}}$.

$$QCoh(S^1_{Fil}) = \lim_{\Delta^{op}} D(H^\bullet) \qquad \heartsuit = \lim_{\Delta^{op}} (\heartsuit) = H\text{-rep}_{\mathbb{Z}_{(p)}}.$$

$$\geq 0 \qquad \neq \lim_{\Delta^{op}} D^{\geq 0}(H^\bullet)$$

$$\leq 0 \qquad = \lim_{\Delta^{op}} D^{\leq 0}(H^\bullet)$$

$$B\mathbb{G}_a \rightsquigarrow \underline{\underline{R\Gamma(B\mathbb{G}_a, \mathcal{O})}} \xrightarrow{\mathrm{Spec}?} B\mathbb{G}_a.$$

$$\cdots \quad \mathbb{G}_a^2 \rightrightarrows \mathbb{G}_a \rightrightarrows * \longrightarrow B\mathbb{G}_a$$

$$\mathbb{Z} \rightrightarrows \mathbb{Z}[T_1] \rightrightarrows \mathbb{Z}[T_1, T_2] \rightrightarrows \cdots$$

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}[T_1] \longrightarrow \mathbb{Z}[T_1, T_2] \longrightarrow \cdots$$

deg 1: $\mathbb{Z}[-1] \rightsquigarrow R\Gamma(B\mathbb{G}_a, \mathcal{O}) = L\mathrm{Sym}_{\mathbb{Z}}(\mathbb{Z}[-1])$.

$$\mathrm{Map}_{\mathrm{coSCR}_{\mathbb{Z}}}\left(\mathrm{LSym}_{\mathbb{Z}}(\mathbb{Z}[-1]),\ R\right)$$

$$= \mathrm{Map}_{D(\mathbb{Z})}\left(\mathbb{Z}[-1], R\right) = R[1].$$